Mathematical Foundations of Infinite-Dimensional Statistical Models

4.4 Gaussian and Empirical Processes in Besov Spaces

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Introduction

- Study the connection between certain Gaussian and empirical processes and the Besov spaces $B^s_{pq}([0,1]), s \in \mathbb{R}$
- Throughout this section, we let $\{\psi_{lk}\}$ be an S-regular, S sufficiently large wavelet basis of $L^2([0,1])$.
  - periodised basis
  - boundary-corrected basis
- For convention, the scaling functions ϕ_k equal the 'first' wavelets $\psi_{J-1,k}$, where$J = 0$and $J \in \mathbb{N}$ large enough in the boundary-corrected case, and we recall that there are 2^l wavelets ψ_{lk} at level $l \geq 0$

Gaussian white noise and Brownian motion

- $\mathbb{W}$: Gaussian white noise of isonormal Gaussian process on $L^2([0,1])$

$$\mathbb{W}(g) \sim N(0, ||g||_2^2), \quad E\mathbb{W}(g)\mathbb{W}(g') =< g, g' >, \quad g, g' \in L^2([0,1])$$

- Any ortho-normal basis of L^2 $\{\psi_{lk}\}$ generates an infinite sequence of standard Gaussian r.v.s $g_{lk} = \mathbb{W}(\psi_k) \sim N(0,1)$
- The process $\mathbb{W}$ can be viewed as a generalised function (or element of S^*) simply by considering the action of the random wavelet series

$$\sum_{l \geq J-1} \sum_k g_k \psi_k$$

  on test functions.
- This r.v.s is an element of some B^s_{pq}?
  Equal to check convergence of the Besov sequence norms of (g_{lk}).
- A similar question can be asked for the Brownian bridge process

$$\mathbb{G}(g) \sim N(0, \left\| g - \int_0^1 g \right\|_2^2), \; E\mathbb{G}(g)\mathbb{G}(g') =< g, g' > - \int_0^1 g \int_0^1 g'$$

Proposition 4.4.1

- The white noise process $\mathbb{W}$ and the Brownian bridge process $\mathbb{G}$ define tight Gaussian Borel random variables in $B_{pp}^{-s}([0,1])$ for any $s > 1/2$ and $1 \leq p < \infty$.

- pf) For $e_p = E|g_1 1|^p$, from Fubini's theorem,

$$E \left\| \mathbb{W} \right\|_{B_{pp}^{-s}}^p = \sum_l 2^{pl(-s+1/2-1/p)} \sum_k E|g_{lk}|^p = e_p \sum_l 2^{pl(1/2-s)} < \infty$$

  so $\mathbb{W} \in B_{pp}^{-s}$ almost surely, measurable for the cylindrical σ-algebra. Since B_{pp}^{-s} is separable and complete, mW is Borel measurable and the result follows from the Oxtoby-Ulam theorem. The Brownian bridge case is the same.

Logarithmic Besov spaces

- Logarithmic Besov spaces

$$B_{pp}^{s,\delta} \equiv \left\{ f : \|f\|_{B_{pp}^{s,\delta}}^p \equiv \sum_l 2^{pl(s+1/2-1/p)} \max(l,1)^{p\delta} \sum_k |\langle \psi_{lk}, f\rangle|^p < \infty \right\}, \delta, s \in \mathbb{R}$$

- Note that $B_{pp}^{s,0} = B_{pp}^s$, but otherwise we can decrease or increase the regularity of the functional space on the logarithmic scale.

- **Proposition 4.4.2** The white noise process $\mathbb{W}$ and the Brownian bridge process $\mathbb{G}$ define tight Gaussian Borel r.v.s in $B_{pp}^{-1/2,-\delta}([0,1])$ for any $1 \leq p < \infty, \delta > 1/p$.

Proposition 4.4.3

- For any $1 \le p < \infty$, the random variables $\|\mathbb{W}\|_{B^{-1/2}_{p\infty}([0,1])}$ and $\|\mathbb{G}\|_{B^{-1/2}_{p\infty}([0,1])}$ are finite almost surely.

- pf) For every M large enough and $e_p = E\,|g_{11}|^p$, from a union bound and chebyshev's inequality,

$$\Pr\left(\|\mathbb{W}\|_{B^{-1/2}_{p\infty}>M}\right) = \Pr\left(\sup_l 2^{-l}\sum_k |g_{lk}|^p > M^p\right)$$

$$\le \sum_l \Pr\left(2^{-l}\sum_k (|g_{kk}|^p - e_p) > M^p - e_p\right) \le \frac{1}{(M^p - e_p)^2}\sum_l 2^{-l} e_{2p}$$

so for M large enough, we deduce

$$\Pr\left(\|\mathbb{W}\|_{B^{-1/2}_{p\infty}} < \infty\right) > 0$$

(0-1 law for Gaussian measures) + (Besov norm, countable supremum of finite-dimensional ℓ_p-norms, is measurable for the cylindrical σ-algebra $\mathcal{C}$). The Brownian bridge case is again the same.

Other cases

- Borell-Sudakov-Tsirelson inequality implies the random variables

$$\|\mathbb{W}\|_{B^{-1/2}_{p\infty}([0,1])}, \quad \|\mathbb{G}\|_{B^{-1/2}_{p\infty}([0,1])}$$

  are actually sub-Gaussian.

- If $\max(p,q) < \infty$, $\mathbb{W}, \mathbb{G}$ are not tight in $B^{-1/2}_{p\infty}$(nonseparable).

$p = q = \infty$

- **Theorem 4.4.4** (a) For $\omega = (\omega_l) = (\sqrt{l})$, we have

$$\Pr\left(\|\mathbb{W}\|_{B^{-1/2,\omega}_{\infty\infty}(0,1])} < \infty\right) = 1$$

(b) For any w s.t. $\left(w_l/\sqrt{l}\right) \uparrow \infty$ as $l \to \infty$, the white noise process $\mathbb{W}$ defines a tight Gaussian Borel r.v. in the closed subspace $B^{-1/2,w}_{\infty\infty\infty,0}$ of $B^{-1/2,w}_{\infty\infty}$ consisting of coefficient sequences satisfying

$$\lim_{l\to\infty} w_l^{-1} \max_k |\langle f, \psi_{lk}\rangle| = 0$$

(c) The preceding statements remain true if $\mathbb{W}$ is replaced by $\mathbb{G}$.

Donsker Properties of Balls in Besov Spaces

- P: prob. measure on A. $\mathcal{B}$: subset of a Besov space $B^s_{pq}(A)$.
- Question: $\mathcal{B}$ is P-pre-Gaussian or even P-Donsker?
    - $A = [0, 1]$ case.
- Certain Besov balls will be shown to be P-pre-Gaussian but not P-Donsker.

Besov Balls with $s > 1/2$

- **Theorem 4.4.5** Let $1 \leq p, q \leq \infty$, and assume that $s > max(1/p, 1/2)$. Then any bounded subset $\mathcal{B}$ of $B^s_{pq}([0,1])$ is a uniform Donsker class. In particular, bounded subsets of Sobolev spaces $H^s([0,1])$ and Holder spaces $C^s([0,1])$ are P-Donsker for $s > 1/2$ and any P.

- To be precise, since we require $s > 1/p$, we can and do view as $\mathcal{B}$ as a family of conti. functions in the preceding theorem. this result implies in particular that $\mathcal{B}$ is P-pre-Gaussian for any P.

Besov Balls with $s > 1/2$

- **Proposition 4.4.6** If P has a bounded Lebesgue density on $[0, 1]$, then any bounded subset $\mathcal{B}$ of $B^s_{pq}([0, 1])$ for $1 \leq p, q \leq \infty$ and $s > 1/2$ is P-pre-Gaussian.

- An interesting gap between Thm 4.4.5 and Prop. 4.4.6 arises when $1 \leq p < 2$ and P indeed has a bounded density.
- This gap provides examples for P-pre-Gaussian classes of functions that are not P-Donsker.

- **Proposition 4.4.7** Suppose that P has a bounded Lebesgue density on $[0, 1]$, and let $1/2 < s1$. The unit ball $\mathcal{B}$ of $B^{\mathrm{s}}_{1\infty}([0, 1])$ is P-pre-Gaussian but not P-Donsker.

Besov Balls with $s > 1/2$

- **Remark 4.4.8** $B^{s}_{1\infty}([0,1])$ consist of not necessarily conti. functions and hence has to be viewed as a space Lebesgue-equivalence class of functions. Empirical processes are not defined on equivalence classes of functions but on functions.
  The set of all a.e. modifications of a fixed function can easily be shown not to be P-Donsker, so to avoid triviality, the preceding statement should be understood as holding for $\mathcal{B}$ equal to any class of functions constructed from selecting one element f from each equivalence class $[f]$ in the unit ball of $B^{s}_{1\infty}([0,1])$.

Donsker Properties for Critical Values of s

- **Proposition 4.4.9** Bounded subsets of $B^{1/p}_{p1}(A), 1 \leq p < 2$, are uniform Donsker classes for A any interval in (possibly equal to) $\mathbb{R}$.

Donsker Properties for Critical Values of s

- **Theorem 4.4.10** For $\delta > 1/2$, any bounded subset $\mathcal{B}$ of $B_{22}^{1/2,\delta}([0,1])$ consists of uniformly bounded continuous functions and is P-Donsker for any P with bounded Lebesgue density on $[0,1]$.